ON BOUNDED DISTORTIONS OF MAPS IN THE LINE

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Abstract. We give an example illustrating that two notions of bounded distortion for $\mathcal{C}^1$ expanding maps in $\mathbb{R}$ are different.

1. Introduction and definitions

Let I_1 and I_2 be disjoint closed intervals and let $F : I_1 \cup I_2 \to [0,1]$ be a $\mathcal{C}^1$ map such that $F|_{I_i}$ is a diffeomorphism on $[0,1]$ and $F' > 1$ on its domain. The map F has associated a unique repeller K given by $K = \cap_{k\geq 1} F^{-k}([0,1])$; the set K is the maximal invariant set under F and is a Cantor set. Its Hausdorff and upper box dimensions coincide, and they may be equal to 1 (see [PT93], Chapter 4). More information on K is obtained imposing conditions on the map F. More precisely, let Ω_k be the set of words of length k with symbols 0 and 1, and note that the k-th iterate F^k is defined on the family of 2^k closed intervals $\{I_\omega : \omega \in \Omega_k\}$, labeled from left to right using the lexicographical order on Ω_k. Note that the restriction $F^k|_{I_\omega}$ is a diffeomorphism onto $[0,1]$. We say that the map F satisfies the *bounded distortion property* **BD** if there exists a constant $1 \leq C < \infty$ such that

$$\frac{(F^k)'(x)}{(F^k)'(y)} \leq C \quad \textit{for all } k > 0,$$

and for all $x, y \in I_\omega$ and $\omega \in \Omega_k$. Moreover, F satisfies the *strong bounded distortion property* **SBD** if there is a sequence β_l decreasing to 1 such that

$$\frac{(F^k)'(x)}{(F^k)'(y)} \leq \beta_r \quad \textit{for all } k > 0,$$

whenever x, y belong to the same basic interval $I_\omega, \omega \in \Omega_k$ and $|F^k([x,y])| \leq 1/r$, where $|A|$ denotes the diameter of the set A.

Clearly **SBD** implies **BD**. Moreover, it is well known that if F' is α-Hölder continuous, then **SBD** holds (see for example [PT93]), and the same is true if the modulus of continuity $w(t) = \sup_{|x-y|<t} |F'(x) - F'(y)|$ satisfies the Dini condition $\int_0^1 w(t)t^{-1}dt$ (see [FJ99]). Let $\dim_H K$ denotes the Hausdorff dimension of K. Property **BD** implies that $0 < \dim_H K < 1$, and also that the $\dim_H K$-dimensional Hausdorff measure is positive and finite. Moreover, property **SBD** is needed, for example, to define the scaling function, which is a $\mathcal{C}^1$ complete invariant for Cantor sets defined by smooth maps: two such sets with the same scaling function are diffeomorphic (see [BF97]).

However, although one suspects that **SBD** is actually stronger than **BD**, we did not find in the literature an example illustrating this fact. The purpose of this note is to provide such an example.

2. The example

In order to construct F, we need a special family $\{\varphi_t\}_{t\in[-1,1]}$ of smooth diffeomorphisms of the interval $[0,1]$. For this reason, let X be the C^∞ field on $[0,1]$ defined by $X(0) = X(1) = 0$ and $X(x) = \exp((x(x-1))^{-1})$. Consider its associated flow $\{\varphi_t\}_{t\in\mathbb{R}}$ (see for example [Lan02]): for each $x \in [0,1]$ let $\tilde\phi(t,x)$ be the solution of the equation

$$\begin{cases} \frac{d}{dt}\phi(t,x) = X(\phi) \\ \phi(0,x) = x \end{cases};$$

then $\varphi_t = \tilde\phi(t,\cdot)$. Note $\tilde\phi \in \mathcal{C}^\infty(\mathbb{R}\times[0,1])$, which by the initial condition implies $\varphi_t(0) = 0$ and $\varphi_t(1) = 1$ for all t. Below we list the properties of $\{\varphi_t\}_{t\in[-1,1]}$ that we will use:

i) $\varphi_0(x) = x$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$, whenever $t, s, t+s \in [-1,1]$;
ii) $\varphi_t'(0) = \varphi_t'(1) = 1$, for all t;
iii) $\|\varphi_t' - 1\|_u \to 0$ as $t \to 0$;
iv) $\phi_t'(x) \geq 2/3$, for all x and $t \in [-T,T]$, for some $0 < T \leq 1$;
v) there exists M such that $\|\varphi_t''\|_u \leq M$ for all $t \in [-1,1]$.

Property i) is the semigroup property for flows; ii) follows from the identity $(\partial/\partial t)(\partial/\partial x)\tilde\phi(t,x) = X'(\tilde\phi(x,t))(\partial/\partial x)\tilde\phi(t,x)$ and since $X'(0) = X'(1) = 0$; iii) and v) are consequence of the smoothness of $\tilde\phi$, while iv) follows from iii).

For $n \geq 0$, let $J_n = [2/3^{n+1}, 1/3^n]$ and denote by $A_n : J_n \to [0,1]$ and $B_n : [0,1] \to J_{n-1}$ the affine maps

$$A_n(x) = 3^{n+1}x - 2 \quad and \quad B_n(x) = \frac{x+2}{3^n}.$$

Note

$$A_n \circ B_{n+1} = id_{[0,1]}. \tag{1}$$

Also, for $2^k \leq n < 2^{k+1}$, let $t_n = (-1/2)^k T$. We define

$$F : [0,1/3] \cup [2/3,1] \to [0,1]$$

by

$$F(x) = \begin{cases} B_n \circ \varphi_{t_n} \circ A_n(x), & if\ x \in J_n,\ n \geq 1 \\ 3x, & if\ x \in [0,1/3] \setminus \cup_{n\geq 1} J_n \\ 3x-2, & if\ x \in [2/3,1] \end{cases}.$$

Note that F satisfies the diagram below.

Lemma 1. *The function F is $\mathcal{C}^1$.*

$$\begin{tikzcd} J_n \arrow[r, "F"] \arrow[d, "A_n"'] & J_{n-1} \\ {[0,1]} \arrow[r, "\varphi_{t_n}"'] & {[0,1]} \arrow[u, "B_n"'] \end{tikzcd}$$

Proof. Clearly F' exists and is continuous on $\big((0,1/3]\setminus\cup_{n\geq 1}J_n\big)\cup[2/3,1]$. The existence and continuity of F' at $1/3^n$ and $2/3^n$ follows computing the left and right-sided derivatives and using ii): both are equal to 3. Moreover, for the right hand sided derivative at 0, if $h\in J_n$, then by the mean value theorem,

$$\left|\frac{F(h)-F(0)}{h}-3\right| = \left|3\varphi'_{t_m}(3^{m+1}\xi_h-2)-3\right| = 3\left|\varphi'_{t_m}(3^{m+1}\xi_h-2)-1\right|$$

(for some $m\geq n$ and $\xi_h\in J_m$), which tends to 0 by iii). This implies the existence of $F'(0)$ (right sided), and the continuity at 0 also follows from iii). $\square$

Given $\omega=\omega_1\ldots\omega_n$ and $\tau=\tau_1\ldots\tau_k$ we define $\omega\tau=\omega=\omega_1\ldots\omega_n\tau_1\ldots\tau_k$. Also, let $0^n,1^n\in\Omega_n$ be the words formed only by zeroes and ones, respectively.

We need two preliminary lemmas.

Lemma 2. *Let* $\tau\in\Omega_k$.

a) $I_{0^n1}=J_n$ *for all* $n>0$. *In particular,* $I_{0^n1\tau}\subset J_n$. *Moreover, if* $x\in I_{0^n1\tau}$ *and* $t=t_1+\cdots+t_n$, *then* $t\in[0,T]$ *and*

$$(F^n)'(x)=3^n\varphi'_t(A_n(x)).$$

b) $I_{1^n\tau}\subset J_0$ *for all* $n>0$. *Moreover, if* $x\in I_{1^n\tau}$, *then*

$$(F^n)'(x)=3^n.$$

Proof. We first notice that $\sum_{2^k\leq n<2^{k+1}}t_n=2^k\cdot(-1/2)^kT=(-1)^kT,\forall k\geq 0$, and so, if $2^k\leq n<2^{k+1}$, k even, then $t_1+\cdots+t_n=T-T+\cdots+T-T+\sum_{2^k\leq m\leq n}(-1/2)^kT=\sum_{2^k\leq m\leq n}(-1/2)^kT=\frac{n-2^k+1}{2^k}T\in[0,T]$, and if $2^k\leq n<2^{k+1}$, k odd, then $t_1+\cdots+t_n=T-T+\cdots+T-T+T+\sum_{2^k\leq m\leq n}(-1/2)^kT=T-\sum_{2^k\leq m\leq n}(1/2)^kT=(1-\frac{n-2^k+1}{2^k})T\in[0,T]$.

By definition, F is a bijection from J_n to J_{n-1}. Therefore, it can be shown inductively that $I_{0^n1}=J_n$, for all $n>0$. In particular, if $x\in J_n$ we have by i) and (1) that

$$F^n(x)=B_1\circ\varphi_t\circ A_n(x), \tag{2}$$

and differentiating we obtain part *a*). Part *b*) is immediate from the definition of F. $\square$

The following is an estimate on the size of basic intervals.

Lemma 3. *For each $n \geq 0$ and $\tau \in \Omega_k$ we have*

$$|I_{0^n 1\tau}| \leq 3^{-n+1} 2^{-k-2}.$$

Proof. Denote by $f_{0^n 1\tau}$ the inverse of $F^{k+n}|_{I_{0^n 1\tau}}$, which is a diffeomorphism onto $[0,1]$. Then, for some $\xi \in (0,1)$ we have

$$|I_{0^n 1\tau}| = f'_{0^n 1\tau}(\xi) = \frac{1}{(F^{k+n+1})'(f_{0^n 1\tau}(\xi))}.$$

Set $y := f_{0^n 1\tau}(\xi) \in I_{0^n 1\tau}$. Then, by Lemma 2 *a*) and iv) we have

$$\begin{aligned}(F^{k+n+1})'(y) &= (F^{k+1})'(F^n(y))(F^n)'(y)\\ &= 3^n \varphi'_{t_1+\cdots+t_n}(A_n(y))(F^{k+1})'(F^n(y))\\ &\geq \frac{2}{3} 3^n 2^{k+1},\end{aligned}$$

and the lemma follows. □

Now we are ready to verify that F satisfies **BD** but not **SBD**.

F does not satisfies **SBD**. Let $\alpha, \beta \in [0,1]$ be such that $\varphi'_T(\alpha) \neq \varphi'_T(\beta)$ (they exist since $\varphi_T \neq Id$). Observe that for each k we have

$$F^{2^k}|_{J_{2^{k+1}-1}} = B_{2^k} \circ \varphi_{(-1)^k T} \circ A_{2^{k+1}-1}.$$

Then, if k is even and if $x, y \in J_{2^{k+1}-1}$ are such that $A_{2^{k+1}-1}(x) = \alpha$ and $A_{2^{k+1}-1}(y) = \beta$, we obtain

$$\frac{(F^{2^k})'(x)}{(F^{2^k})'(y)} = \frac{\varphi'_T(A_{2^{k+1}-1}(x))}{\varphi'_T(A_{2^{k+1}-1}(y))} = \frac{\varphi'_T(\alpha)}{\varphi'_T(\beta)} \neq 1,$$

whence **SBD** does not hold; indeed, $F^{2^k}(J_{2^{k+1}-1}) = J_{2^k-1}$, whose size tends to 0 when $k \to \infty$.

F satisfies **BD**. Fix $k > 0$ and $\omega \in \Omega_k$ and let $x, y \in I_\omega$. We consider the blocks of zeroes and ones of ω, that is, there is an $L > 0$ such that $\omega = 0^{m_1} 1^{n_1} 0^{m_2} \cdots 0^{m_L} 1^{n_L}$, where $m_j, n_j > 0$ for all j but possibly $m_1 = 0$ or $n_L = 0$ (the case in which ω begins with 1 or ends with 0, respectively). We have $F^k(x) = F^{n_L} \circ F^{m_L} \circ \cdots \circ F^{n_1} \circ F^{m_1}(x)$. Then for each j, F^{m_j} is evaluated at a point $x_j \in I_{0^{m_j}\tau_j} \subset J_{m_j}$, where $|\tau_j| = n_j + \sum_{i=j+1}^{L}(m_i + n_i)$, hence by Lemma 2 *a*),

$$(F^{m_j})'(x_j) = 3^{m_j} \varphi'_{\ell_j}(A_{m_j}(x_j))$$

for some $\ell_j \in [-T, T]$. Moreover, F^{n_j} is evaluated at a point $\tilde{x}_j \in I_{1^{n_j}\gamma}$, where $|\gamma| = \sum_{i=j+1}^{L}(m_i + n_i)$, hence by Lemma 2 *b*), $(F^{n_j})'(\tilde{x}_j) = 3^{n_j}$.

Therefore

$$\begin{aligned}
\frac{(F^k)'(x)}{(F^k)'(y)} &= \prod_{j=1}^{L} \frac{(F^{m_j})'(x_j)}{(F^{m_j})'(y_j)} \\
&= \prod_{j=1}^{L} \frac{\varphi'_{\ell_j}(A_{m_j}(x_j))}{\varphi'_{\ell_j}(A_{m_j}(y_j))} \\
&= \prod_{j=1}^{L} \left(1 + \frac{\varphi'_{\ell_j}(A_{m_j}(x_j)) - \varphi'_{\ell_j}(A_{m_j}(y_j))}{\varphi'_{\ell_j}(A_{m_j}(y_j))}\right) \\
&= \prod_{j=1}^{L} \left(1 + \frac{\varphi''_{\ell_j}(\xi_j)}{\varphi'_{\ell_j}(A_{m_j}(y_j))} 3^{m_j+1}(x_j - y_j)\right) \\
&\leq \prod_{j=1}^{L} \left(1 + \frac{3^3 M}{2^{|\tau_j|+2}}\right),
\end{aligned}$$

the inequality follows from iv), v) and Lemma 3, since $|x_j - y_j| \leq |I_{0^{m_j}\tau_j}|$. The last product is uniformly bounded since $\sum_{j=1}^{L} 2^{-|\tau_j|} \leq \sum_{i=0}^{\infty} 2^{-i} < \infty$. Therefore F satisfies **BD**.

Acknowledgments

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